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A Comparison of Pseudo-Maximum Likelihood and Asymptotically Distribution-Free Dynamic Factor Analysis Parameter Estimation in Fitting Covariance-Structure Models to Block-Toeplitz Matrices Representing Single-Subject Multivariate Time-Series

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The study of intraindividual variability pervades empirical inquiry in virtually all subdisciplines of psychology. The statistical analysis of multivariate time-series data — a central product of intraindividual investigations — requires special modeling techniques. The dynamic factor model (*DFM*), which is a generalization of the traditional common factor model, has been proposed by Molenaar (1985) for systematically extracting information from multivariate time-series via latent variable modeling. Implementation of the *DFM* model has taken several forms, one of which involves specifying it as a covariance-structure model and estimating its parameters from a block-Toeplitz matrix derived from the multivariate time-series. We compare two methods for estimating *DFM* parameters within a covariance-structure framework — pseudo-Maximum Likelihood (*p-ML*) and Asymptotically Distribution Free (*ADF*) estimation — by means of a Monte Carlo simulation. Both methods appear to give consistent model parameter estimates of comparable precision, but only the *ADF* method gives standard errors and chi-square statistics that appear to be consistent. The relative ordering of the values of all estimates appears to be very similar across methods. When the manifest time-series is relatively short, the two methods appear to perform about equally well.

From clinical process through signal analysis, from the study of group dynamics and memory to the modeling of stage-wise development, multivariate time-series reflecting intraindividual variability are natural constituents of empirical research. Primarily because a time-series is composed of dependent observations, its statistical analysis requires the use of special modeling techniques. Dynamic factor analysis (*DFA*) (Molenaar, 1985; Wood & Brown, 1994), a primary focus of this article, is one of the

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techniques developed specifically for modeling single-subject, multivariate time-series data. The dynamic factor model (*DFM*), which is a generalization of the traditional common factor model, requires the use of specialized procedures for its implementation.

The Dynamic Factor Model

In its purest form *DFA* is factor analysis of a multivariate time-series obtained by the repeated measurement of one subject. Hence, in contrast to standard factor analysis which focuses on interindividual covariation, *DFA* is concerned with the structuring of intraindividual covariation. Analyses of the structure of intraindividual covariation can yield results that differ substantially from analogous interindividual structural analysis outcomes. For instance, finding a one-factor model in standard longitudinal factor analysis of interindividual covariation is quite compatible with the finding that the intraindividual covariation of none of the subjects constituting the longitudinal sample obeys a dynamic one-factor model. Examples of such discrepancies between the structure of inter- ~~and intra-individual~~ covariation have been given with simulated data (Molenaar, 1997). Moreover, substantial evidence of individual differences in the structure of intraindividual covariation has been established in the mood-emotion literature (cf. Hershberger, Corneal, & Molenaar, 1994; Molenaar, Rovine & Corneal, in press; Rovine, Molenaar, & Corneal, in press; also Shifrin, Hooker, Wood, & Nesselroade, 1997). Nesselroade and Molenaar (in press) reported similar individual differences in the intraindividual structure of cognitive and biomedical time series data obtained with multiple elderly subjects. Such findings heavily underscore the need for analyses of intraindividual covariation in many contexts, for example, predicting individual developmental trajectories, counseling, and individual psychotherapy.

Apart from its intraindividual focus and the consequent technical requirements to handle sequentially dependent time-series data, the basic tenets of *DFA* and standard factor analysis are the same. This is immediately evident if we recognize *DFA* as a direct generalization of *P*-technique factor analysis (Cattell, 1963), that is, the straightforward application of standard factor analysis to time series data. The *DFA* model reduces to *P*-technique if the observed series lacks sequential dependency (Jones & Nesselroade, 1990, Molenaar, 1985; and cf. Nesselroade & Cattell, 1988, for further discussion of the rationale and many applications of *P*-technique).

Moving in the other direction, several degrees of generalization of *P*-technique can be distinguished. The *DFA* model introduced by Molenaar (1985) is based upon the assumption that the observed multivariate time

series is weakly stationary. This implies that the observed series lacks a deterministic trend and that the structure of sequential covariation is homogeneous across time. The assumption of weak stationarity in *DFA* is comparable to the standard factor analysis assumption that the population of subjects is independent and identically distributed (a formal specification of weak stationarity is given in the next section). A further generalization is obtained by foregoing the assumption that the observed series lacks a deterministic trend, yielding a *DFM* with nonstationary mean trend and stationary sequential covariance (Molenaar, deGooijer, & Schmitz, 1992). The most far-reaching generalization is obtained by dropping the assumption of weak stationarity completely, yielding a *DFM* with nonstationary mean trend and sequential covariance. For the latter completely nonstationary *DFM*, (Molenaar, 1994b) presented an expectation-maximization (*EM*) estimation method that makes use of the Kalman filter. The Kalman filter serves as a recursive estimator of the latent factor series in the expectation step of the *EM* algorithm (See Shumway, 1988, for an excellent, concise introduction to Kalman filtering).

In this article we will restrict attention to the *DFA* model for weakly stationary Gaussian multivariate time-series presented in Molenaar (1985). There are three principal reasons for doing so. First, the statistical theory of weakly stationary time series is highly developed (Hannan, 1970) and constitutes the foundation of time-series analysis in general. In what follows we will make use of a key result from this powerful theory. Second, the assumption of weak stationarity will hold for many natural (behavioral) processes if the length of the total observation interval is sufficiently bounded with respect to the intrinsic rate of change of the processes under scrutiny. Moreover, nonstationary mean trends and most forms of nonstationary sequential covariance can be removed by data transformation. The frequency domain (Fourier transformed) analog of the *DFM* for weakly stationary series has been successfully applied in many other fields of science, including engineering (Brillinger, 1975; Priestley & Subba Rao, 1973), econometrics (Engle & Watson, 1981; Geweke & Singleton, 1981), and psychophysiology (Molenaar, 1987; 1994a; Nunez, 1981). These applications in the frequency domain, however, require a relatively large number of repeated measurements (at least a few hundred), whereas such extensive data are not required for the time domain analog (Molenaar, 1985). Third and last, parameter estimation in the latter model can be carried out by means of standard structural equation modeling software that is readily accessible to social scientists.

Estimation Issues

We will compare two statistical methods for fitting the *DFM* to multivariate time-series. First, the class of so-called weakly stationary time-series to which we restrict our attention will be characterized. Next, a concise description of *DFM*s for weakly stationary multivariate time-series will be presented and alternative techniques for fitting a *DFM* will be identified. We then provide a detailed consideration of one particular technique in which the *DFM* is reformulated as a covariance-structure model. The covariance-structure model has to be fitted to a block-Toeplitz matrix, that is, a patterned covariance matrix characterizing the observed multivariate time-series. The statistical estimation can be accomplished by the method of maximum likelihood, but with a caveat. On the one hand, the method of maximum likelihood requires that the vector-valued observations be independent. On the other hand, the estimate of the block-Toeplitz matrix that is analyzed is based on dependent multivariate time-series data. Hence we refer to the application of the method of maximum likelihood estimation to such data as the pseudo-maximum likelihood (pseudo-*ML*) method.

One key reason to use the pseudo-*ML* method to estimate *DFM* parameters is that it can be applied straightforwardly, solely using commercially available software. Only the estimation of the block-Toeplitz matrix requires special software, but such is available from Wood and Brown (1994) or else one can implement his or her own procedure ~~relatively easily~~.[1] Another practical reason for using the pseudo-*ML* approach is that it allows one to fit the *DFM* to a p-variate time-series when p is relatively large.

The statistical properties of the pseudo-*ML* fit of *DFM*s, reformulated as covariance-structure models, to block-Toeplitz matrices characterizing multivariate time-series are largely unknown. Nor, to the best of our knowledge, has the matter been addressed by means of Monte Carlo techniques. Below, we present the results of a limited Monte Carlo study that does so. To broaden the scope of our investigation, the pseudo-*ML* method will be compared with the asymptotically distribution-free (*ADF*) estimation method (Browne, 1984).

With the *ADF* method a covariance-structure embodiment of a *DFM* also is fitted to a block-Toeplitz matrix. The *ADF* method is a weighted least-squares procedure in which the weight matrix has to be specified properly in order to guarantee that the asymptotic properties of standard normal theory estimators and test statistics are obtained (Bentler & Dudgeon, 1996; Browne & Shapiro, 1988). In the present context, this implies that the proper weight

[1] Our implementation of a method for estimating the block-Toeplitz matrices can be found at (ftp://kiptron.psyc.virginia.edu/ftp/pub/jrn).

matrix for a block-Toeplitz matrix characterizing a multivariate observed series has to be determined. It is here that a powerful result of the statistical theory of weakly stationary time-series comes into play. The elements of a block-Toeplitz matrix are lagged auto- and cross-covariances (see the next section for further specification), where each element is estimated from dependent observations (i.e., the multivariate time-series data). Under the assumption that this series is weakly stationary and Gaussian, it is known that the asymptotic properties of a block-Toeplitz matrix thus estimated closely correspond to those of standard normal theory estimators. More specifically, an estimated block-Toeplitz matrix obtained with sequentially dependent time-series data is asymptotically unbiased and has a normal sampling distribution with known covariance (Anderson, 1971, p. 481). The covariance of the latter sampling distribution, however, differs from the one obtained in standard normal theory in that it is a function of the sequential dependency of the observations. Hence, if the weight matrix in *ADF* estimation is constructed by using the proper covariance of the sampling distribution of a block-Toeplitz matrix based on sequentially dependent time-series data, the asymptotic properties of standard normal theory estimators and test statistics are guaranteed. In this sense, then, the *ADF* method corrects for the sequential dependency of the observations. To produce the weight matrix in the *ADF* method, we have developed a special technique which will be described subsequently.[2]

The order of the block-Toeplitz matrix is $wp \times wp$, where w is an integer, $w > 0$ (see the explanation of block-Toeplitz matrices given below), and p is the dimension of the observed time-series. The order of the weight matrix in the *ADF* method is $[wp(wp + 1)/2] \times [wp(wp + 1)/2]$ which quickly becomes very large as p increases. Hence the *ADF* method is practical for *DFA* of p-variate series only when p is relatively small.

Modeling Concerns

Before comparing estimation procedures, there are several definitions and specifications to be made concerning models. These include the concepts of *stationarity*, especially weak stationarity, the *DFM* specification, and some additional discussion of the *ADF* estimation method.

Stationarity

The following notational conventions will be used: vector-valued variables are denoted by bold-face lower-case letters, matrix-valued variables

[2] Our implementation of this technique can be found at (ftp://kiptron.psyc.virginia.edu/ftp/pub/jrn).

by bold-face upper-case letters. Roman letters are used for manifest variables, Greek letters for latent variables. The subscripted symbol Σ denotes summation, with the subscript indicating the index variable. The superscript $'$ denotes matrix transposition.

A stochastic process $\mathbf{z}(t)$ in discrete time t is characterized by an ensemble of finite-dimensional distributions $P(\mathbf{z}; t) = Prob[\mathbf{z}(t) < \mathbf{z}]$, $P(\mathbf{z}_1, \mathbf{z}_2; t_1, t_2) = Prob[\mathbf{z}(t_1) < \mathbf{z}_1; \mathbf{z}(t_2) < \mathbf{z}_2]$, etcetera. Accordingly, $\mathbf{z}(t)$ can be regarded as a random time-dependent function and we can consider its first-order moment function, second-order moment function, etcetera. In general, these moment functions can be time-varying. If, however, the first-order moment function is a constant, $E[\mathbf{z}(t)] = \mathbf{c}_z$, then $\mathbf{z}(t)$ is called first-order stationary. If its second-order central moment function only depends upon the lag between t_1 and t_2,

$$E\{[\mathbf{z}(t_1) - \mathbf{c}_z(t_1)], [\mathbf{z}(t_2) - \mathbf{c}_z(t_2)]'\} = cov[\mathbf{z}(t_1), \mathbf{z}(t_2)'] = \mathbf{C}_z(k),\ k = t_2 - t_1,$$

then $\mathbf{z}(t)$ is called second-order stationary. If $\mathbf{z}(t)$ is both first- and second-order stationary then it is called weakly stationary.

Dynamic Factor Models for Weakly Stationary Multivariate Series

Definition and Explanation of the Dynamic Factor Model

The *DFM* (Molenaar, 1985, 1994; Wood & Brown, 1994; see also McArdle, 1982) combines two important analytical tools — multivariate time-series models and the common factor model. Motivating *DFM*'s development was recognition of the value of factor analyzing multivariate time-series (Bereiter, 1963; Cattell, 1963) coupled with the realization that, without suitable modification, the traditional common factor model did not fully exploit the information inherent in multivariate time-series. Indeed, applying the common factor model to time-series data could be misleading with certain kinds of process information (Holtzman, 1963; Molenaar, 1985; Steyer, Ferring, & Schmitt, 1992).

The *DFM* incorporating q factors and s lags of manifest variables on common factors (*DFM* [q, s]) is specified as:

$\Lambda(s)\cdot\eta(t-s)$

$$\mathbf{z}(t) = \Lambda(0)\,\boldsymbol{\eta}(t) + \Lambda(1)\,\boldsymbol{\eta}(t-1) + \ldots + \Lambda(s-1)\,\boldsymbol{\eta}(t-s+1) + \boldsymbol{\varepsilon}(t),$$

where $\mathbf{z}(t)$ is the observed or manifest p-variate time-series, $\boldsymbol{\eta}(t)$ is the latent q-variate factor time-series, $\boldsymbol{\varepsilon}(t)$ is a p-variate noise time-series, and $\Lambda(u)$, $u = 0, 1, \ldots, s-1$, are $p \times q$ matrices of lagged factor loadings. Thus,

the various $\Lambda(u)$, $u = 0, 1, \ldots, s-1$, can differ from each other, signifying that the regressions of the variables on the common factors vary as a function of the amount of lag. Said another way, the effects of the factors on the manifest variables can decay differentially. Fitting the *DFM* model to manifest time-series involves the estimation of $\Lambda(u)$, $u = 0, 1, \ldots, s-1$, and the covariance matrices of $\boldsymbol{\eta}(t)$ and of $\boldsymbol{\varepsilon}(t)$.

Identifiability Of Covariance Function Latent Factor Series

It can be shown, however, that for unconstrained $\Lambda(u)$, $u = 0, 1, \ldots, s$, where $s > 0$, the covariance function $\mathbf{C}_\eta(k)$ of $\boldsymbol{\eta}(t)$ is not identified (Molenaar, 1985) and hence should be fixed. A convenient choice is: $\mathbf{C}_\eta(k) = \mathbf{I}_q\,\delta(k)$, where $\mathbf{I}_q$ denotes the $(q \times q)$-dimensional identity matrix, $\delta(k) = 1$ if $k = 0$ and $\delta(k) = 0$, otherwise. However, any other choice of fixed positive-definite block-Toeplitz matrix is acceptable in this situation. Beyond this special case, a general approach to ascertain the identifiability of *DFM*s was given by Molenaar (1989) and consists of the following steps: (a) Fourier transformation of a *DFA* model, yielding a frequency-dependent set of static, complex-valued factor models; (b) reformulation of each model in this set into its real-valued analog; and (c) application of commercially available algebraic software for exact evaluation of identifiability in static, restricted factor models (Bekker, Merckens, & Wansbeek, 1994; for more details see Molenaar, 1989; for more details about dynamic factor analysis in the frequency domain see Molenaar, 1987).

Relationship With State-Space Models

If $s = 0$ then the covariance function of $\boldsymbol{\eta}(t)$ can be identifiable with unconstrained loadings (Molenaar, 1985). In this case the standard *DFM* reduces to a generalized state-space model in which $\boldsymbol{\eta}(t)$ is an identifiable q-variate autoregressive-moving average (cf. Molenaar, de Gooijer, & Schmitz, 1992, for a general approach to testing identifiability in *DFM*s).

Estimation of Dynamic Factor Models

Reformulation of the Dynamic Factor Model as a Covariance Structure Model

The *DFM* can be formulated as a covariance structure model by (a) defining a symmetric covariance matrix that incorporates the lagged covariance structure of the manifest multivariate time-series and (b) specifying the model parameter constraints in a manner that is consistent with both the *DFM* and the redundancies that are, of necessity, incorporated into the symmetric covariance matrix to which the model is fitted. We discuss these two points in order.

As before, let $\mathbf{z}(t)$, $t = 1, 2, 3, ..., T$, denote a p-variate time-series (T occasions in length). Let $\mathbf{C}_z(u)$ denote the $p \times p$ matrix-valued covariance function of $\mathbf{z}(t)$ at lag u ($u = 0,1,2,..., w$). Thus, $\mathbf{C}_z(0)$ is the $p \times p$ covariance matrix with no (zero) lagging of the variables on themselves or each other. It is the matrix one would obtain by treating the n occasions as though they were n cases and covarying the p variables across them. This matrix is the one factored in traditional P-technique factor analysis (see e.g., Cattell, 1963). $\mathbf{C}_z(1)$ is the $p \times p$ matrix of lag 1 covariances between variables (i.e., at time t and time $t + 1$). This matrix, which although square is ordinarily not symmetric, contains the lag 1 autocovariances of the variables in its principal diagonal. In a similar manner, the remaining $\mathbf{C}_z(u)$ up to some maximum lag ($u = w$) are constructed.

The various $\mathbf{C}_z(u)$ are put together to form a block-Toeplitz covariance matrix, $\mathbf{S}_z(w)$, as follows:

$$\mathbf{S}_z(w) = \mathbf{C}_z(j - k);\ j, k = 0, 1, ..., w$$

with $\mathbf{C}_z(-u) = \mathbf{C}'_z(u)$. Thus, $\mathbf{S}_z(w)$ is a supermatrix of order $p(w + 1) \times p(w + 1)$, the submatrices of which are the unlagged and lagged covariance matrices. For example, if $p = 6$ and $w = 4$ (signifying lags of 0, 1, 2, 3, and 4), the corresponding block-Toeplitz matrix is 30×30. The construction of a block-Toeplitz matrix for $w = 4$ is illustrated in Figure 1. It contains $w + 1$ lag-0 portions (symmetric), w lag-1 portions (asymmetric), $w - 1$ lag-2 portions (asymmetric), ..., and 1 lag-4 portion (asymmetric). The diagonal entries of all submatrices represent autocovariances of the variables for the corresponding number of lags. Although it is highly redundant, the block-Toeplitz matrix is a symmetric covariance matrix to which the *DFM* can be fitted. For additional discussion of the nature of these block-Toeplitz matrices see Wood and Brown (1994).

An important consideration is the choice of the value of the maximum lag, w. Molenaar, de Gooijer, and Schmitz (1992) gave a principled method for estimating the lower bound for w prior to any application of the *DFA* model. Their method involves fitting univariate parametric time-series models (subject to a simple constraint) to each of the component series of an observed p-variate series to determine the smallest allowable value of w. (For a complete exposition of the method, which can be carried out with commercially available software, the reader is referred to Molenaar et al., 1992, p. 339). A more informal, approximate approach consists of plotting

$$\begin{array}{|c|c|c|c|c|}
\hline
\mathbf{C_z}(0) & & & & \\
\hline
\mathbf{C_z}(1) & \mathbf{C_z}(0) & & & \\
\hline
\mathbf{C_z}(2) & \mathbf{C_z}(1) & \mathbf{C_z}(0) & & \\
\hline
\mathbf{C_z}(3) & \mathbf{C_z}(2) & \mathbf{C_z}(1) & \mathbf{C_z}(0) & \\
\hline
\mathbf{C_z}(4) & \mathbf{C_z}(3) & \mathbf{C_z}(2) & \mathbf{C_z}(1) & \mathbf{C_z}(0) \\
\hline
\end{array}$$

Figure 1
Toeplitz-Transformed Covariance Matrix For Lags 0, 1, 2, 3, and 4.

the auto- and cross-covariance functions of the component series of the manifest p-variate series to determine the lag at which these all become negligible (the latter can be substantiated by means of appropriate significance tests). Due to a theorem in the statistical theory of stationary time-series that auto- and cross-covariance functions decay to zero at an exponential rate (e.g., Hannan, 1970, p. 151), the lower bound on admissible values of w can be expected to occur at a finite lag.

The DFM that can be fitted to the block-Toeplitz can be specified as follows: $\mathbf{\Lambda}$, the matrix of factor loadings, is of order $p(w + 1) \times q(2w + 1)$, where q = the number of factors. $\mathbf{\Lambda}$ consists of $(w + 1)\ p \times q$ blocks of lag-0 loadings, $(w + 1)\ p \times q$ blocks of lag-1 loadings, $(w + 1)\ p \times q$ blocks of lag-2 loadings, etcetera, arrayed as shown in Figure 2 (next page).

The factor covariance matrix is symmetric and of order $q(2w + 1)$. It also contains redundant submatrices, as indicated in Figure 2.

The analog of the covariance matrix of the unique parts is also portrayed in Figure 2. It has the same block-Toeplitz pattern as $\mathbf{S}_z(w)$, but now all blocks

Λ, Factor Loading Matrix

Λ(0)	Λ(1)	Λ(2)	Λ(3)	Λ(4)	0	0	0	0
0	Λ(0)	Λ(1)	Λ(2)	Λ(3)	Λ(4)	0	0	0
0	0	Λ(0)	Λ(1)	Λ(2)	Λ(3)	Λ(4)	0	0
0	0	0	Λ(0)	Λ(1)	Λ(2)	Λ(3)	Λ(4)	0
0	0	0	0	Λ(0)	Λ(1)	Λ(2)	Λ(3)	Λ(4)

E, Uniqueness Covariance Matrix Analog

diag[C_ε(0)]	diag[C_ε(1)]	diag[C_ε(2)]	diag[C_ε(3)]	diag[C_ε(4)]
diag[C_ε(1)]	diag[C_ε(0)]	diag[C_ε(1)]	diag[C_ε(2)]	diag[C_ε(3)]
diag[C_ε(2)]	diag[C_ε(1)]	diag[C_ε(0)]	diag[C_ε(1)]	diag[C_ε(2)]
diag[C_ε(3)]	diag[C_ε(2)]	diag[C_ε(1)]	diag[C_ε(0)]	diag[C_ε(1)]
diag[C_ε(4)]	diag[C_ε(3)]	diag[C_ε(2)]	diag[C_ε(1)]	diag[C_ε(0)]

Φ, Factor Covariance Matrix

♦	0	0	0	0	0	0	0	0
0	♦	0	0	0	0	0	0	0
0	0	♦	0	0	0	0	0	0
0	0	0	♦	0	0	0	0	0
0	0	0	0	♦	0	0	0	0
0	0	0	0	0	♦	0	0	0
0	0	0	0	0	0	♦	0	0
0	0	0	0	0	0	0	♦	0
0	0	0	0	0	0	0	0	♦

Figure 2
Factor Model Matrices When Dynamic Factor Model is Fitted as a Covariance Structure Model. For unconstrained lagged factor loadings the **Φ** submatrices are fixed at the identity matrix.

are diagonal $p \times p$ dimensional matrices $\mathbf{C}_\varepsilon(j\text{-}k)$; j, $k = 0, 1, \ldots, w$, where $\mathbf{C}_\varepsilon(u) = \text{diag}[c_{\varepsilon_1}(u), \ldots, c_{\varepsilon_p}(u)]$.

By specifying this model and introducing equality constraints that reflect the redundancies that have been indicated, the parameters can be estimated via commercially available software such as LISREL 8 (Jöreskog & Sörbom, 1994).

ADF Method in Covariance Structure Modeling of Block-Toeplitz Matrices

The ADF method is a weighted least-squares method in which the weight matrix consists of the asymptotic covariances associated with the estimated covariance matrix of the data (Browne, 1984; Jöreskog & Sörbom, 1994). If the estimated covariance matrix is a block-Toeplitz matrix $\mathbf{S}_z(w)$ of a single realization of a p-variate time-series $\mathbf{z}(t)$, $t = 1,2,\ldots,T$, then the asymptotic covariances concerned are associated with the estimated covariance function $\mathbf{C}_z(u)$ making up the blocks of $\mathbf{S}_z(w)$. Let $c_{ij}(m)$ and $c_{kl}(n)$ denote two arbitrary elements of $\mathbf{C}_z(u)$, $|u| = 0, 1, \ldots, w$ [and hence of $\mathbf{S}_z(w)$], where i, j, k, l take values in $(1, 2, \ldots, p)$ and m, n take values in $(-w, \ldots, w)$. In addition let $\mathbf{z}(t)$ be Gaussian, let $\mathrm{E}[\mathbf{z}(t)] = 0$, and denote the expected values of elements of $\mathbf{C}_z(u)$ by $E[c_{ij}(u)] = \gamma_{ij}(u)$. Then the asymptotic covariance of $c_{ij}(m)$ and $c_{kl}(n)$ is given by (cf. Hannan, 1970, p. 209):

$$\text{cov}\,[c_{ij}(m), c_{kl}(n)] = T^{-1}\Sigma_u(1-|u|/T)\,[\gamma_{ik}(u)\,\gamma_{jl}(u+n-m) + \gamma_{il}(u+n)\,\gamma_{jk}(u-m)]$$

where the index u for the summation $\mathbf{\Sigma}_u$ runs from $-T+1$ to $T-1$.

The asymptotic covariances depend upon the expected values of the covariance function $\mathbf{C}(u)$ at lags $|u|$ of the order of T. Because the $\gamma_{ij}(u)$ in the expression for the asymptotic covariances have to be replaced by their estimates $c_{ij}(u)$ based on a single realization $\mathbf{z}(t)$, $t = 1,2,\ldots,T$, we need a special approach to obtain these estimates at large lags $|u|$ approaching T. A convenient way to obtain $c_{ij}(u)$ estimates at all lags $|u| = 0, 1, \ldots, T-1$, is to fit a p-variate autoregression to $\mathbf{z}(t)$ and then derive the required $c_{ij}(u)$ from this autoregression. The theoretical underpinning of this approach is given by Wold's decomposition theorem (cf. Hannan, 1970, p. 137), according to which each weakly stationary time-series can be represented by an autoregression. To see how this works, suppose that $\mathbf{z}(t)$ obeys a first-order autoregression:

$$\mathbf{z}(t) = \mathbf{A}\,\mathbf{z}(t-1) + \boldsymbol{\xi}(t),$$

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where **A** is a $p \times p$ matrix of autoregression coefficients and $\boldsymbol{\xi}(t)$ is a p-variate white noise series. Then $\mathrm{cov}[\mathbf{z}(t), \mathbf{z}(t+u)] = \mathbf{C}(u) = \mathbf{A}^u$, $u > 0$. Hence the entire covariance function of this first-order autoregression is obtained as a simple function of the single matrix **A** of autoregression coefficients. Generalization to the case of an r^{th} order autoregression, $r > 1$, is straightforward (cf. Kashyap & Rao, 1976).

Our approach to determine the weight matrix in the *ADF* method for the analysis of block-Toeplitz matrices can now be summarized as follows. First, an autoregression is fitted to the observed p-variate series $\mathbf{z}(t)$, $t = 1, 2, ..., T$. We use the algorithm of Jones (1978), in which the order of the autoregression is determined by means of Akaike's information criterion. The autoregression thus obtained is used to estimate all the $c_{ij}(u)$ that are required as proxies for the $\gamma_{ij}(u)$ in the expression for the asymptotic covariances making up the *ADF* weight matrix.[3] It is noted that because of the redundancy of a block-Toeplitz matrix, the associated weight matrix will in general be singular. A ridge option involving the substitution of a small positive constant for zero eigenvalues is used to obtain a nonsingular weight matrix.

A Comparison of Pseudo-ML and ADF Methods

To recapitulate, the pseudo-*ML* method involves the use of the *ML* technique in fitting a *DFM*, reformulated as a covariance-structure model, to a block-Toeplitz matrix that has been estimated from dependent observations (i.e., observed multivariate time-series data). As was alluded to earlier, there do not seem to exist published studies in which the statistical properties of the pseudo-*ML* method applied to time-series data have been evaluated. We will present the results of a small Monte Carlo study comparing the performance of the pseudo-*ML* and *ADF* estimation methods.

Simulation Model

Independent realizations of a 3-variate weakly stationary time-series $\mathbf{z}(t)' = [\mathbf{z}_1(t), \mathbf{z}_2(t), \mathbf{z}_3(t)]$ were generated according to the following dynamic 1-factor model:

[3] A listing of the Fortran subroutine concerned can be found at (ftp://kiptron.psyc.virginia.edu/ftp/pub/jrn).

$$\mathbf{z}(t) = \Lambda(0)\,\boldsymbol{\eta}(t) + \Lambda(1)\,\boldsymbol{\eta}(t-1) + \Lambda(2)\,\boldsymbol{\eta}(t-2) + \varepsilon(t), \text{ where,}$$

$\Lambda(0)' = [1.0, 0.9, 0.0]$
$\Lambda(1)' = [0.7, 0.6, 0.6]$
$\Lambda(2)' = [0.4, 0.3, 0.3]$
$\boldsymbol{\eta}(t-s) \sim$ i.i.d. $N(0,1)$, $s = 0, 1, 2$; where $N(0,1)$ refers to the normal distribution with mean 0 and variance 1.0.
$\varepsilon_i(t) = 0.5\,\varepsilon_i(t-1) + a_i(t)$, $a_i(t) \sim$ i.i.d. $N(0,1)$, $i = 1, 2, 3$; where $N(0,1)$ refers to the normal distribution with mean 0 and variance 1.0.

Notice that at lag zero the loading of the univariate latent factor series $\boldsymbol{\eta}(t)$ on $z_3(t)$ is zero, whereas these loadings on $z_1(t)$ and $z_2(t)$ are nonzero. Under this dynamic 1-factor model the expected covariance functions between $z_3(t)$ and $z_1(t)$ and between $z_3(t)$ and $z_2(t)$ are asymmetric. Notice also that the measurement errors $\varepsilon_i(t)$, $i = 1, 2, 3$, are autocorrelated (but the cross-correlations between $\varepsilon_i(t)$ and $\varepsilon_j(t)$, for $i \neq j$, are zero). Hence, this dynamic 1-factor model constitutes a typical instance of the class of dynamic 1-factor models with arbitrary lead-lag relationships between the latent factor series and the manifest series and with arbitrarily autocorrelated measurement errors.

Three sets of 50 independent realizations were generated according to this dynamic 1-factor model. In the first set the length of each 3-variate series was $T = 50$, in the second set T = 100, and in the third set $T = 500$. We will refer to these three sets as conditions 1, 2, and 3, respectively.

Design

Within each condition, each of the 50 realizations was subjected to a dynamic 1-factor analysis by means of the pseudo-*ML* method and the *ADF* method. That is, the dynamic 1-factor model used to generate the data was rewritten as a covariance-structure model and fitted to the block-Toeplitz matrix estimated from each given realization. The window width for determining block-Toeplitz matrices was $w = 2$, hence their order was 9×9 $[p(w+1) \times p(w+1)]$. Accordingly, for each realization in each condition two sets of parameter estimates were obtained, one by means of the pseudo-*ML* method and another by means of the *ADF* method. Because these two sets of parameter estimates were obtained with the same set of 50 block-Toeplitz matrices, they can be compared directly.

Results

For each of the three conditions we will present the results obtained with application of both the pseudo-*ML* and the *ADF* methods to the 50 independent replications of an estimated block-Toeplitz matrix. First, we present the means of the pseudo-*ML* and *ADF* parameter estimates and compare these with the true parameter values used in generating the data. Second, the means of the associated standard errors estimated by means of both methods are presented. These mean standard errors are compared with the standard deviations of the parameter estimates across the 50 replications. Third, the pseudo-*ML* and *ADF* parameter estimates and their estimated standard errors are compared in a straightforward correlational analysis. Fourth, and finally, we tested whether or not the chi-square goodness-of-fit statistics obtained with both methods follow a chi-square distribution (with degrees of freedom to be estimated) and the correlation between pseudo-*ML* and *ADF* chi-square statistics is given.

Condition 1: T = 50

The mean pseudo-*ML* and *ADF* estimates of the lagged factor loadings are presented in Table 1, together with the associated true values. Notice that the following convention has been used in Table 1: $\Lambda(u)' = [\lambda_1(u), \lambda_2(u), \lambda_3(u)]$, $u = 0, 1, 2$, where the vectors of lagged loadings $\Lambda(u)$ belong to the dynamic 1-factor model specified above. The standard deviation of the mean *p-ML* estimates about the true values of the lagged loadings is .11, while this standard deviation is .12 for the mean *ADF* estimates.

Next we turn to the estimated autocovariance functions of the measurement errors. From the dynamic 1-factor model used to generate the data the true autocovariance functions of the measurement errors $\varepsilon_i(t)$, $i = 1, 2, 3$, can be derived. Each univariate measurement error series obeys the same first-order autoregression model which is repeated here for convenience: $\varepsilon_i(t) = \text{phi}^* \varepsilon_i(t) + a_i(t)$, phi = .5 and $a_i(t) \sim$ i.i.d. $N(0,1)$. It then follows that $\text{cov}[\varepsilon_i(t), \varepsilon_i(t+u)] = c_i(u)$ is given by: $c_i(u) = c_i(0)^*(\text{phi}^u)$, $u = 0, 1, \ldots$, where the autocovariance $c_i(0)$ is given by $c_i(0) = \text{var}[a_i(t)] / (1.0 - \text{phi}^2)$, where $\text{var}[a_i(t)] = 1.0$. Substituting the value of phi then yields: $c_i(0) = 1.333$, $c_i(1) = .66$, $c_i(2) = .33$, for $i = 1, 2, 3$. Table 2 presents the true values as well as their pseudo-*ML* and *ADF* mean estimates of the measurement error autocovariance functions. The standard deviation of the mean pseudo-*ML* estimates about the true values of the measurement errorautocovariance function is .21, whereas this standard deviation for the mean *ADF* estimates is .23.

Table 1
True and Mean Estimated Lagged Factor Loadings in Dynamic 1-factor Model Obtained with 50 Independent Realizations of a 3-variate Manifest Series of Length $T = 50$ According to the Pseudo-Maximum Likelihood (*p-ML*) and Asymptotically Distribution-Free (*ADF*) Estimation Methods.

Loading	true	*p-ML*	*ADF*
$\lambda_1(0)$	1.000	.957	.915
$\lambda_2(0)$	.900	.929	.911
$\lambda_3(0)$	.000	.082	.068
$\lambda_1(1)$	.700	.546	.514
$\lambda_2(1)$	.600	.462	.428
$\lambda_3(1)$	.600	.580	.523
$\lambda_1(2)$	.400	.370	.325
$\lambda_2(2)$	.300	.278	.238
$\lambda_3(2)$	.300	.538	.499

Table 2
True and Mean Estimated Autocovariance Functions (acov) of the Measurement Errors in a Dynamic 1-factor Model Obtaincd with 50 Independent Realizations of a 3-variate Manifest Series of Length $T = 50$ According to the Pseudo-Maximum Likelihood (*p-ML*) and Asymptotically Distribution-Free (*ADF*) Estimation Methods.

acov	true	*p-ML*	*ADF*
$c_1(0)$	1.330	1.455	1.346
$c_2(0)$	1.330	1.111	1.042
$c_3(0)$	1.330	.909	.876
$c_1(1)$	.660	.805	.697
$c_2(1)$	.660	.567	.510
$c_3(1)$	.660	.312	.272
$c_1(2)$	.330	.404	.352
$c_2(2)$	.330	.303	.271
$c_3(2)$	.330	.231	.196

The estimated standard errors associated with the parameter estimates are presented in some detail for reasons to be given in the closing section. Table 3 gives the mean standard errors for estimated lagged factor loadings obtained with the pseudo-*ML* and *ADF* methods. Also presented are the standard deviations of the parameter estimates obtained with the pseudo-*ML* and *ADF* methods across the 50 independent replications.

Using the same notation as in Table 3, the mean standard errors and the standard deviations of the sampling distributions of estimated autocovariance functions of the measurement errors are presented in Table 4.

As to the chi-square statistics obtained with both methods, their means are 4.013 for the pseudo-*ML* method and 1.783 for the *ADF* method. Following the rule given in Molenaar (1985), the degrees of freedom of the block-Toeplitz matrices in the covariance-structure analyses are $df = 24$ (6 *df* for $\mathbf{C}_z(0)$ and 9 *df* for $\mathbf{C}_z(1)$ and $\mathbf{C}_z(2)$ each). The number of free parameters

Table 3
Mean Standard Errors (s.e.) and Standard Deviations (s.d.) of the Empirical Sampling Distributions of the Estimated Lagged Loadings in the Dynamic 1-factor Model, Obtained with 50 Independent Replications of a 3-variate Manifest Series of Length $T = 50$ According to the Pseudo-Maximum Likelihood (*p-ML*) and Asymptotically Distribution-Free (*ADF*) Estimation Methods. Lagged Factor Loadings are Indexed as in Table 1.

Loading	s.e.(*p-ML*)	s.d.(*p-ML*)	s.e.(*ADF*)	s.d.(*ADF*)
$\lambda_1(0)$	.202	.248	.307	.246
$\lambda_2(0)$	.188	.272	.287	.265
$\lambda_3(0)$	.192	.240	.271	.239
$\lambda_1(1)$	.188	.258	.293	.254
$\lambda_2(1)$	.176	.253	.274	.231
$\lambda_3(1)$	.151	.218	.224	.197
$\lambda_1(2)$	.256	.305	.365	.272
$\lambda_2(2)$	.243	.307	.346	.271
$\lambda_3(2)$	.209	.259	.298	.253
column mean	.201	.262	.296	.248

Table 4
Mean Standard Errors (s.e.) and Standard Deviations (s.d.) of the Empirical Sampling Distributions of the Estimated Autocovariance Functions of the Measurement Errors in the Dynamic 1-factor Model, Obtained with 50 Independent Replications of a 3-variate Manifest Series of Length $T = 50$ According to the Pseudo-Maximum Likelihood (*p-ML*) and Asymptotically Distribution-Free (*ADF*) Estimation Methods. Notation is as in Table 2.

acov	s.e.(*p-ML*)	s.d.(*p-ML*)	s.e.(*ADF*)	s.d.(*ADF*)
$c_1(0)$	.462	.511	.725	.499
$c_2(0)$	.372	.436	.581	.431
$c_3(0)$	.270	.351	.384	.333
$c_1(1)$	.333	.401	.545	.380
$c_2(1)$	.258	.387	.415	.337
$c_3(1)$	.191	.298	.280	.273
$c_1(2)$	.350	.360	.475	.312
$c_2(2)$	.285	.281	.381	.231
$c_3(2)$	.236	.204	.292	.185
column mean	.306	.359	.453	.331

in the dynamic 1-factor model is 18, hence the chi-square goodness-of-fit statistics have 6 degrees of freedom. The mean of the chi-square distribution which these statistics are assumed to follow asymptotically therefore is 6.

Within each condition (e.g., pseudo-*ML*, $NT = 50$) 50 chi-square goodness-of-fit values were obtained. The question is whether or not these 50 values can be considered to be realizations of a chi-square distribution, where the degrees of freedom of this chi-square distribution constitutes the free (to be estimated) parameter. For each of the three sets, this proposition was tested by means of the Pearson chi-square goodness-of-fit test. To implement the test, guidelines given by Moore (1986) were followed. In particular, the cells in this test were chosen to have equal probabilities under the hypothesized chi-square distribution and the number $M = 9$ of cells was chosen according to the criterion given by Moore (1986, p. 70). The unknown degrees of freedom parameter of the hypothesized chi-square distribution was estimated. Then, the Pearson-Fisher statistic follows a chi-square distribution with $df = M - 2 = 7$ under the null hypothesis, no matter the true value of the unknown degrees of freedom parameter of the hypothesized chi-square distribution (cf. Moore, 1986, p. 66). The following results were obtained for the $T = 50$ condition:[4]

p-ML: estimated $df = 4.013$, Pearson's statistic = 5.440, $df = 7$, $p = .606$
ADF: estimated $df = 1.783$, Pearson's statistic = 9.400, $df = 7$, $p = .225$

Thus, the set of 50 chi-squared goodness-of-fit values obtained in the $T = 50$ condition can be considered to be a realization of a chi-squared distribution with the estimated *df* concerned.

The estimated lagged factor loadings obtained via the pseudo-*ML* and *ADF* methods are correlated .97. The analogous correlation for the estimated measurement error autocovariances is also .97. The estimated standard errors of the lagged factor loadings obtained with both methods are correlated .93, while this correlation is .94 for the standard errors of the autocovariance functions of the measurement errors. Finally, the chi-square goodness-of-fit statistics obtained with the pseudo-*ML* and *ADF* methods are correlated .97.

Condition 2: T = 100

Next, the number of realizations (observations) for each of the 50 independent replications was increased from 50 to 100. The estimated lagged loadings obtained with the pseudo-*ML* and *ADF* methods are given in Table 5. The standard deviation of the mean *p-ML* estimates about the true values of the lagged loadings is .08 and this standard deviation is also .08 for the mean *ADF* estimates.

The true and estimated autocovariance functions of the measurement errors are presented in Table 6. The standard deviation of the mean pseudo-*ML* estimates about the true values of the measurement error autocovariance function is .16, whereas this standard deviation for the mean *ADF* estimates is .21.

In Table 7 are given the mean standard errors and empirical standard deviations of the lagged factor loadings estimated by the pseudo-*ML* and *ADF* methods.

Table 8 gives the mean standard errors and empirical standard deviations of the estimated measurement error covariance functions obtained with the pseudo-*ML* and *ADF* methods.

As to the chi-square statistics obtained with both methods, their means are 6.046 for the pseudo-*ML* method and 2.920 for the *ADF* method. As was

[4] This work was carried out by Dr. Hilde Huizenga.

Table 5
True and Mean Estimated Lagged Factor Loadings in Dynamic 1-factor Model Obtained with 50 Independent Realizations of a 3-variate Manifest Series of Length $T = 100$ According to the Pseudo-Maximum Likelihood (*p-ML*) and Asymptotically Distribution-Free (*ADF*) Estimation Methods.

Loading	true	*p-ML*	*ADF*
$\lambda_1(0)$	1.000	.987	.980
$\lambda_2(0)$	.900	.961	.936
$\lambda_3(0)$	.000	.004	-.003
$\lambda_1(1)$	.700	.680	.622
$\lambda_2(1)$	.600	.530	.519
$\lambda_3(1)$	.600	.522	.461
$\lambda_1(2)$	.400	.408	.388
$\lambda_2(2)$	.300	.344	.317
$\lambda_3(2)$	.300	.509	.472

Table 6
True and Mean Estimated Autocovariance Functions (acov) of the Measurement Errors in a Dynamic 1-factor Model Obtained with 50 Independent Realizations of a 3-variate Manifest Series of Length $T = 100$ According to the Pseudo-Maximum Likelihood (*p-ML*) and Asymptotically Distribution-Free (*ADF*) Estimation Methods.

acov	true	*p-ML*	*ADF*
$c_1(0)$	1.330	1.241	1.128
$c_2(0)$	1.330	1.059	.996
$c_3(0)$	1.330	1.071	1.025
$c_1(1)$	.660	.606	.531
$c_2(1)$	.660	.533	.457
$c_3(1)$	.660	.466	.422
$c_1(2)$	.330	.328	.274
$c_2(2)$	.330	.197	.193
$c_3(2)$	.330	.277	.242

Table 7
Mean Standard Errors (s.e.) and Standard Deviations (s.d.) of the Empirical Sampling Distributions of the Estimated Lagged Loadings in the Dynamic 1-factor Model, Obtained with 50 Independent Replications of a 3-variate Manifest Series of Length T = 100 According to the Pseudo-Maximum Likelihood (*p-ML*) and Asymptotically Distribution-Free *ADF* Methods. Lagged Factor Loadings are Indexed as in Table 5.

Loading	s.e.(*p-ML*)	s.d.(*p-ML*)	s.e.(*ADF*)	s.d.(*ADF*)
$\lambda_1(0)$	.143	.188	.220	.186
$\lambda_2(0)$	.138	.169	.213	.152
$\lambda_3(0)$	.133	.174	.190	.172
$\lambda_1(1)$	.127	.171	.203	.166
$\lambda_2(1)$	.123	.190	.194	.167
$\lambda_3(1)$	.105	.178	.150	.162
$\lambda_1(2)$	.188	.233	.270	.224
$\lambda_2(2)$	.177	.244	.256	.219
$\lambda_3(2)$	.152	.232	.213	.229
column mean	.143	.198	.212	.186

Table 8
Mean Standard Errors (s.e.) and Standard Deviations (s.d.) of the Empirical Sampling Distributions of the Estimated Autocovariance Functions of the Measurement Errors in the Dynamic 1-factor Model, Obtained with 50 Independent Replications of a 3-variate Manifest Series of Length T = 100 According to the Pseudo-Maximum Likelihood (*p-ML*) and Asymptotically Distribution-Free (*ADF*) Estimation Methods. Notation is as in Table 6.

acov	s.e.(*p-ML*)	s.d.(*p-ML*)	s.e.(*ADF*)	s.d.(*ADF*)
$c_1(0)$	.326	.449	.533	.408
$c_2(0)$	.283	.364	.468	.383
$c_3(0)$	.179	.232	.258	.206
$c_1(1)$	.229	.348	.378	.324
$c_2(1)$	.188	.307	.323	.282
$c_3(1)$	.132	.215	.197	.195
$c_1(2)$	.230	.272	.311	.236
$c_2(2)$	.199	.243	.272	.211
$c_3(0)$	.166	.187	.206	.172
column mean	.215	.291	.327	.269

alluded to earlier, the mean of the chi-square distribution that these statistics are assumed to follow asymptotically is 6. The 50 chi-square goodness-of-fit values obtained under the T = 100 condition were tested to determine if they could be considered to be realizations of a chi-square distribution where the degrees of freedom constitutes the free (to be estimated) parameter. The following results were obtained for the T = 100 condition:

p-ML: estimated df = 6.046, Pearson's statistic = 6.160, df = 7, p = .521
ADF: estimated df = 2.920, Pearson's statistic = 10.480, df = 7, p = .163

Thus, the set of 50 chi-squared goodness-of-fit values obtained in the T = 100 condition appear to be a realization of a chi-squared distribution with the estimated df concerned.

The estimated lagged factor loadings obtained with the two methods are correlated .90. The analogous correlation for the estimated measurement error autocovariances is .95. The estimated standard errors of the lagged factor loadings obtained with both methods are correlated .83, while this correlation is .90 for the standard errors of the autocovariance functions of the measurement errors. Finally, the chi-square goodness-of-fit statistics obtained with the pseudo-*ML* and *ADF* methods are correlated .92.

Condition 3: T = 500

Table 9 (next page) gives the estimated lagged factor loadings obtained with the pseudo-*ML* and the *ADF* methods for the 50 independent realizations with T = 500. The standard deviation of the mean *p-ML* estimates about the true values of the lagged loadings is .012 and this standard deviation is .005 for the mean *ADF* estimates.

The true and estimated autocovariance functions of the measurement errors are presented in Table 10. The standard deviation of the mean pseudo-*ML* estimates about the true values of the measurement error autocovariance function is .047, while this standard deviation for the mean *ADF* estimates is .049.

The mean standard errors and empirical standard deviations of the lagged factor loadings estimated by the pseudo-*ML* and *ADF* methods are presented in Table 11.

Table 12 gives the mean standard errors and empirical standard deviations of the estimated measurement error covariance functions obtained with the pseudo-*ML* and *ADF* methods.

The means of the chi-square statistics were 8.957 for the pseudo-*ML* method and 5.403 for the *ADF* method. As alluded to earlier, the mean of the chi-square distribution that these statistics are assumed to follow

Table 9

True and Mean Estimated Lagged Factor Loadings in Dynamic 1-factor Model Obtained with 50 Independent Realizations of a 3-variate Manifest Series of Length $T = 500$ According to the Pseudo-Maximum Likelihood (*p-ML*) and Asymptotically Distribution-Free (*ADF*) Estimation Methods.

Loading	true	*p-ML*	*ADF*
$\lambda_1(0)$	1.000	.994	.989
$\lambda_2(0)$	.900	.909	.898
$\lambda_3(0)$	.000	.004	-.002
$\lambda_1(1)$	.700	.707	.696
$\lambda_2(1)$	.600	.612	.603
$\lambda_3(1)$	.600	.612	.599
$\lambda_1(2)$	.400	.417	.399
$\lambda_2(2)$	.300	.318	.305
$\lambda_3(2)$	.300	.316	.307

Table 10

True and Mean Estimated Autocovariance Functions (acov) of the Measurement Errors in a Dynamic 1-factor Model Obtained with 50 Independent Realizations of a 3-variate Manifest Series of Length $T = 500$ According to the Pseudo-Maximum Likelihood (*p-ML*) and Asymptotically Distribution-Free (*ADF*) Estimation Methods.

acov	true	*p-ML*	*ADF*
$c_1(0)$	1.330	1.308	1.284
$c_2(0)$	1.330	1.414	1.385
$c_3(0)$	1.330	1.250	1.230
$c_1(1)$	.660	.648	.631
$c_2(1)$	.660	.715	.690
$c_3(1)$	.660	.625	.610
$c_1(2)$	.330	.328	.319
$c_2(2)$	.330	.342	.329
$c_3(2)$	.330	.298	.287

Table 11

Mean Standard Errors (s.e.) and Standard Deviations (s.d.) of the Empirical Sampling Distributions of the Estimated Lagged Loadings in the Dynamic 1-factor Model, Obtained with 50 Independent Replications of a 3-variate Manifest Series of Length $T = 500$ According to the Pseudo-Maximum Likelihood (*p-ML*) and Asymptotically Distribution-Free (*ADF*) Estimation Methods. Lagged Factor Loadings are Indexed as in Table 9.

Loading	s.e.(*p-ML*)	s.d.(*p-ML*)	s.e.(*ADF*)	s.d.(*ADF*)
$\lambda_1(0)$	.063	.103	.091	.099
$\lambda_2(0)$	.058	.087	.085	.089
$\lambda_3(0)$	.055	.090	.078	.091
$\lambda_1(1)$	.062	.097	.098	.091
$\lambda_2(1)$	.057	.104	.091	.100
$\lambda_3(1)$	.048	.073	.068	.069
$\lambda_1(2)$	.093	.133	.123	.129
$\lambda_2(2)$	.087	.117	.115	.109
$\lambda_3(2)$	.083	.116	.110	.113
column mean	.067	.102	.095	.099

Table 12

Mean Standard Errors (s.e.) and Standard Deviations of the Empirical Sampling Distributions (s.d.) of the Estimated Autocovariance Functions of the Measurement Errors in the Dynamic 1-factor Model, Obtained with 50 Independent Replications of a 3-variate Manifest Series of Length $T = 500$ According to the Pseudo-Maximum Likelihood (*p-ML*) and Asymptotically Distribution-Free (*ADF*) Estimation Methods. Notation is as in Table 10.

acov	s.e.(*p-ML*)	s.d.(*p-ML*)	s.e.(*ADF*)	s.d.(*ADF*)
$c_1(0)$	.158	.240	.258	.225
$c_2(0)$	.128	.202	.213	.189
$c_3(0)$	.074	.170	.123	.161
$c_1(1)$	.117	.174	.198	.164
$c_2(1)$	.098	.180	.167	.171
$c_3(1)$	.063	.124	.103	.121
$c_1(2)$	.112	.147	.155	.137
$c_2(2)$	.099	.157	.137	.147
$c_3(2)$	.074	.088	.096	.088
column mean	.103	.165	.161	.159

asymptotically is 6.0. The 50 chi-square goodness-of-fit values obtained under the $T = 500$ condition were tested to determine if they could be considered to be realizations of a chi-square distribution where the degrees of freedom constitutes the free (to be estimated) parameter with the following results:

p-ML: estimated $df = 8.957$, Pearson's statistic = 29.560, $df = 7$, $p = .0001$
ADF: estimated $df = 5.403$, Pearson's statistic = 8.680, $df = 7$, $p = .276$

According to these results, the 50 chi-squared goodness-of-fit values obtained in the *p-ML* $T = 500$ condition cannot be considered to be realizations of a chi-squared distribution with appropriate degrees of freedom. The 50 chi-squared goodness-of-fit values for the *ADF* $T = 500$ condition can be so considered.

The estimated lagged factor loadings obtained by the *p-ML* and *ADF* methods are correlated .96. The analogous correlation for the estimated measurement error autocovariances is .98. The estimated standard errors of the lagged factor loadings obtained with both methods are correlated .91, while this correlation is .92 for the standard errors of the autocovariance functions of the measurement errors. Finally, the chi-square goodness-of-fit statistics obtained with the pseudo-*ML* and *ADF* methods are correlated .96.

Discussion and Conclusion

Before discussing the results of the Monte Carlo-based comparison of the pseudo-*ML* and *ADF* methods for covariance-structure analysis of *DFM*s, we will specify those aspects of the study that restrict the generalizability of the findings. First, only one instance of a *DFM* has been considered, namely a model with a univariate latent factor series and a 3-variate manifest series. Although this model can be regarded as typical of the class of dynamic 1-factor models (involving a lead-lag pattern in factor loadings as well as autocorrelated measurement errors), it is uncertain how the results presently obtained generalize to models having a different number of latent factor and/or manifest series. Second, the number of replications within each condition of our study was limited to 50. On the one hand, this seems sufficient for assessing the bias in parameter estimates and their estimated standard errors but, on the other hand, it is too limited to determine the form of the sampling distributions of parameter estimates and chi-square statistics with sufficient precision. Third, it was our aim in the present simulation study to assess the statistical properties of the pseudo-*ML* and *ADF* methods of dynamic factor analysis in their purest form. To pursue that aim we did not consider the

issues of model selection and modification. That is, the same model used to generate the data also was fitted to these data, thus side-stepping the issue of model selection. Moreover, replications which yielded unacceptable model fits, and hence required modification of the fitted model, were discarded and replaced by additional replications that did yield acceptable fits.

The last point concerning the replacement of replications yielding unacceptable model fits requires further elaboration. It was found that in all cases where an unacceptable model fit was obtained, it was attributable to the estimated autocovariance function of the error series $\varepsilon_i(t)$ being nonpositive-definite (as indicated by a warning in the LISREL output). There were no other causes of an unacceptable model fit, such as lack of convergence of the algorithm minimizing the negative log likelihood ratio. Hence in what follows an unacceptable model fit always concerns an estimated measurement error autocovariance function being nonpositive-definite.

The number of unacceptable model fits was found to depend on the length of the manifest series and the method used to fit the model. In condition 3 ($T = 500$) no unacceptable model fits were observed for either the pseudo-*ML* or *ADF* method. In condition 2 ($T = 100$) it took 67 replications in order to obtain 50 acceptable model fits by means of the pseudo-*ML* method. Of the 17 discarded replications only 3 involved unacceptable model fits by means of the *ADF* method. In condition 1 ($T = 50$) it took 113 replications in order to obtain 50 acceptable model fits by means of the pseudo-*ML* method. Of the 63 discarded replications only 12 involved unacceptable model fits by means of the *ADF* method. Hence, on the basis of these findings, the *ADF* method appears to be superior to the pseudo-*ML* method in that it often yields an acceptable model fit in cases where the pseudo-*ML* method yields a nonpositive-definite estimate of measurement error autocovariance functions.

In summary, the frequency of unacceptable model fits depends upon the length T of a manifest p-variate time-series and upon the method used to fit the model. Our simulation study does not cast light on other factors that may affect the frequency of improper solutions, such as the dimension p of the manifest series, the dimension q of the latent factor series, the maximum lag s of the lagged factor loadings, or the window width w used in constructing a block-Toeplitz matrix. Some additional information can be obtained from the simulation study reported by Wood and Brown (1994) who found, for instance, that the frequency of improper solutions increases as the dimension q of the latent factor series increases. Further study is needed, however, to support a precise evaluation of this issue. If a nonpositive-definite error autocovariance function is obtained in practice, it can be remedied by the introduction of additional constraints, for example, by fixing error autocovariances at the highest lag(s) at zero. As was alluded to earlier, we did not consider such modifications for unacceptable model fits because it was our intention to study the properties of the pseudo-*ML* and *ADF* methods for fitting the *DFM* presented in Molenaar (1985) in their purest forms, unconfounded by issues of model selection, model modification, etcetera. For this reason we also did not consider a new, alternative formulation of the *DFM* with which all unacceptable model fits could have been avoided. As explained in a foregoing section, in the original formulation of the *DFM* as a covariance-structure model the measurement error series $\varepsilon_i(t)$ are represented by their autocovariance functions $c_i(u)$, $u = 0,1,...$. These autocovariance functions are arranged in a block-Toeplitz error covariance structure, where the blocks are diagonal matrices. An alternative way to represent error series $\varepsilon_i(t)$ in a reformulation of the *DFM* as a covariance-structure model is to define the $\varepsilon_i(t)$ as specific latent factor series that are uncorrelated with each other and with the common latent factor series. In this way each univariate measurement error series can be represented by an autoregressive model. It was found in the present simulation study that this alternative representation always yields a positive-definite estimate of the measurement error autocovariance function. However, to reiterate, it is a new formulation that differs substantially from the one originally presented by Molenaar (1985). Its implementation involves technical intricacies that require additional study. Hence, further details of this promising, alternative representation and a simulation study of its statistical properties are delegated to a forthcoming article.

We now turn to a discussion of the results obtained in our Monte Carlo study, bearing in mind the limitations set out above. These results show that the standard deviations of pseudo-*ML* and *ADF* parameter estimates about the true values are almost the same in each condition. Also the absolute values of these standard deviations about the respective true values decrease as the length of the manifest time-series increases. This suggests that parameter estimates obtained with both methods have the same precision and that the pseudo-*ML* and *ADF* methods yield consistent estimates.

The estimated standard errors and the standard deviations of the empirical sampling distributions obtained with both methods present a much more intricate picture. In an attempt to show this picture in a clear way, we presented the results concerned in considerable detail. If we first restrict attention to the results obtained with the pseudo-*ML* method, then it is seen from Tables 3, 4, 7, 8, 11 and 12 that the standard errors are systematically underestimated in comparison with the standard deviations of the empirical sampling distributions. In fact, the differences between the average empirical standard deviations and the average standard errors are consistently about .06 for all parameter estimates across all conditions. Because the absolute values

of both standard errors and empirical standard deviations decrease as the length of the manifest time-series increases, this difference of about .06 in condition 3 ($T = 500$) is of the same order as the absolute values of the standard errors themselves. This result suggests that for relatively long manifest series the pseudo-*ML* method yields estimated standard errors that are biased substantially downward.

The analogous results obtained with the *ADF* method present a different picture. In condition 1 ($T = 50$) the estimated standard errors are larger than the standard deviations of the empirical sampling distributions. The difference between the average standard error and the average empirical standard deviation for lagged factor loading estimates is about .05, whereas this difference is .12 for the measurement error autocovariance estimates. In condition 2 ($T = 100$) the same pattern of results is obtained, although now the absolute value of these differences has decreased (about .03 for lagged factor loadings and about .06 for error autocovariances). But in condition 3 ($T = 500$) the average estimated standard error and the average empirical standard deviations are about the same. This result suggests that for relatively short manifest series the *ADF* method yields estimated standard errors that are biased upwards. In contrast to the pseudo-*ML* method, however, the *ADF* estimates concerned appear to be consistent.

The behavior of the chi-square statistics was anomalous in only one situation — estimation by *p-ML* in the $T = 500$ case. With this exception, the sets of 50 chi-squared goodness-of-fit values can be considered to be realizations of a chi-squared distribution with estimated degrees of freedom appropriate to that case. It appears that the estimated degrees of freedom in conditions 1 and 2 (T relatively small) for the pseudo-*ML* method are close to the nominal value of 6, whereas this is not the case for the *ADF* method. Yet, in condition 3 ($T = 500$), not only is the empirical sampling distribution of the goodness-of-fit values for the pseudo-*ML* method no longer a chi-squared distribution, but the estimated degrees of freedom are too large. In contrast, it appears that the empirical sampling distribution of the goodness-of-fit values for the *ADF* method in condition 3 converges to the chi-square distribution with the nominal degrees of freedom. This result, however, has to be interpreted with caution; a much larger number of replications in each condition than the 50 replications in the present study is needed to arrive at more definite conclusions concerning the empirical sampling distribution of the chi-squared statistic in dynamic factor analysis by means of covariance-structure modeling.

A look at the correlations between the parameter estimates, estimated standard errors and chi-square statistics obtained with the pseudo-*ML* and *ADF* methods clearly shows that these are invariantly high across all conditions.

Apparently the relative order of these estimates in each condition is about the same for each method, while their absolute values may differ by about a constant. This is especially interesting for the chi-square statistics, as it suggests that perhaps model selection based on relative criteria like Akaike's information criterion may yield similar results with both methods. Of course, this is only a suggestion which should be put to test in a Monte Carlo study of the performance of these relative selection criteria under both methods.

As was explained in a former section, the asymptotic covariances of the covariance function estimates of a manifest series were determined by fitting a multivariate autoregressive (MAR) model to the manifest series and then substituting the fitted MAR in the series expansion defining the asymptotic covariances. This yields, after some reordering, the weight matrix in the *ADF* method of LISREL 8. (An equivalent implementation is obtained by using this weight matrix in the *WLS* method of Wood and Brown's (1994) macro.) In case the manifest series is short this procedure will yield a weight matrix of limited precision. Consequently, the limited precision will affect the performance of the *ADF* method. It can be conjectured that this may be the reason for the substantial bias of *ADF* standard error estimates and chi-square statistics in condition 1 and (to a lesser degree) in condition 2. Nevertheless, given the extra demands put on the data by the *ADF* method it is surprising and comforting that the method yields acceptable parameter estimates even with manifest time-series as short as in condition 1 ($T = 50$).

In conclusion, the following points can be drawn from the results of our limited Monte Carlo study of the performance of the pseudo-*ML* and *ADF* methods in covariance-structure modeling of dependent multivariate time-series data. First, both methods appear to give conformable parameter estimates of comparable precision. Second, only the estimated standard errors and chi-square statistics obtained with the *ADF* method appear to be consistent. This assessment is in accord with conclusions drawn by Weng (1990) based on a simulation study of standard (multi-subject) covariance analysis with dependent observations and using a different correction for this dependency. Nevertheless, the results of the simulation, conditional on the qualifications given earlier, corroborate the feasibility of the *p-ML* method, especially in the conditions $T = 50$ and $T = 100$. Third, the relative ordering of the values of all estimates obtained with both methods appears to be about the same. Fourth, and more tentatively, the performance of the pseudo-*ML* method appears to be equivalent in almost all respects to that of the *ADF* method when the length of a manifest series is short. This last point bears directly on the implementation of a proposal for testing the appropriateness of estimating a single block-Toeplitz matrix for dynamic factor analysis from relatively short time-series of multiple participants (Nesselroade & Molenaar, in press).

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